M-math 2nd year Final Exam
Subject : Stochastic Processes

Time : 3.00 hours Max.Marks 50.

1. a) Let $\mathcal{F}_t$ be a filtration and (X_t) be a continuous $\mathcal{F}_t$-adapted, $\mathbb{R}^d$ valued process. Let $F \subset \mathbb{R}^d$ be a closed set. Define

$$\tau := \inf\{s > \sigma : X_s \in F\}$$

where σ is a finite $\mathcal{F}_t$-stopping time. Show that τ is an $\mathcal{F}_t$-stopping time.

b) Show that X_τ is $\mathcal{F}_\tau$ measurable. (7+8)

2. Let (W_t) be a standard 1-dimensional Brownian motion and let for $b > 0$

$$\tau_b := \inf\{s > 0 : W_s = b\}.$$

Show that

$$P\{\tau_b \in A\} = \int_A \frac{b}{\sqrt{2\pi t^3}} exp\{-\frac{b^2}{2t}\}dt$$

for a Borel set $A \subset [0, \infty)$. (8)

3. Let P_0 be the Wiener measure on $C_d[0, \infty)$.
a) Let $n \geq 1, \xi(u) := u^{2n}, u \geq 0$. Find $p : [0, \infty) \to [0, \infty)$, strictly increasing, $p(0) = 0$ such that for $\omega \in C_d[0, \infty)$,

$$B(\omega) := \int_0^T \int_0^T \left\{ \frac{|\omega(t) - \omega(s)|}{p(|t - s|)} \right\}^{2n} dsdt < \infty$$

almost surely P_0.
b) Show that the Brownian paths are Holder continuous of order β for every $\beta, 0 < \beta < \frac{1}{2}$. (8+6)

4. Prove the maximal inequality for a right continuous sub- martingale $\{X_t, \mathcal{F}_t, t \geq 0\}$ i.e. for every $\lambda > 0$,

$$\lambda P\{\sup_{0 \leq t \leq T} X_t > \lambda\} \leq E|X_T|.$$

You may assume the result for discrete time sub-martingales. (10)

5. Let (W_t) be a one dimensional standard Brownian motion. Define

$$Y_t := (1-t)\int_0^t \frac{1}{1-s}dW_s,\ \ 0 \le t < 1$$

and $Y_1 := 0$. Show that $(Y_t, 0 \le t \le 1)$ is a Brownian bridge. (10)